5 Mode, Median, and Mean

Terms: central tendency, mode, median, mean, outlier

Symbols: *Mo*, *Mdn*, *M*, μ , Σ , *N*

Learning Objectives:

- Calculate various measures of central tendency—mode, median, and mean
- Select the appropriate measure of central tendency for data of a given measurement scale and distribution shape
- Know the special characteristics of the mean that make it useful for further statistical calculations

What Is Central Tendency?

You have tabulated your data. You have graphed your data. Now it is time to summarize your data. One type of summary statistic is called central tendency. Another is called dispersion. In this module, we will discuss central tendency.

Measures of **central tendency** are measures of location within a distribution. They summarize, in a single value, the one score that best describes the centrality of the data. Of course, there are lots of scores in any data set. Nevertheless, one score is most representative of the entire set of scores. That's the measure of central tendency. I will discuss three measures of central tendency: the mode, the median, and the mean. Q: How are the mean, median, and mode like a valuable piece of real estate?

A: Location, location, location,

Mode

The mode, symbolized Mo, is the most frequent score. That's it. No calculation is needed.

Here we have the number of items found by 11 children in a scavenger hunt. What was the modal number of items found?

If there are not too many numbers, a simple list of scores will do. However, if there are many scores, you will need to put the scores in order and then create a frequency table. Here are the previous scores in a descending order frequency table.

Score	Frequenc	
20	1	
14	1	
11	2	
8	1	
7	3	
6	1	
5	1	
3	1	

What is the mode? The mode is 7, because there are more 7s than any other number.

Note that the number of scores on either side of the mode does not have to be equal. It might be equal, but it doesn't have to be. In this example, there are three scores below the mode and five scores above the mode.

Nor does the numerical distance of the scores from the mode on either side of the mode have to balance. It could balance, but it doesn't have to balance. Finding the distance of each score from the mode, we get the following values on each side of the mode. As shown below in boldface, -7 does not balance +29.



Distances Below Mode	Distances Above Mode
3 - 7 = -4	8 - 7 = +1
5 – 7 = –2	11 - 7 = +4
6 – 7 = –1	11 - 7 = +4
	14 - 7 = +7
	20 - 7 = +13
$\Sigma = -7$	$\Sigma = +29$



A: He thought it was already mode. The mode is the least stable of the three measures of central tendency. This means that it will probably vary most from one sample to the next. Assume, for example, that we send these same 11 children on another equally difficult scavenger hunt. Now let's assume that every child in this second scavenger hunt finds the same number of items (a very unlikely occurrence in the first place), except that one child who previously found 7 items now finds 11. Compare the two sets of scores below. Only a single score (highlighted in boldface) differs between the two hunts, and yet the mode changes dramatically.

3, 5, 6, 7, 7, 7, 8, 11, 11, 14, 20	first scavenger hunt
3, 5, 6, 7, 7, 11, 8, 11, 11, 14, 20	second scavenger hunt

What is the new mode? It is 11, because there are now three 11s and only two 7s. This is a very big change in the mode, considering that most of the scores in the two hunts were the same. Furthermore, the two hunts would almost certainly be more different than I made them. This, of course, further increases the likelihood that the mode will change.

Because of its simplicity, the mode is an adequate measure of central tendency to report if you need a summary statistic in a hurry. For most purposes, however, the mode is not the best measure of central tendency to report. It is simply too subject to the vagaries of the cases that happen to fall in a particular sample. Also, for very small samples, the mode may have a frequency only one or two higher than the other scores—not very informative. Finally, no additional statistics are based on the mode. For these reasons, it is not as useful as the median or the mean.

Median

The median, symbolized Mdn, is the middle score. It cuts the distribution in half, so that there are the same number of scores above the median as there are below the median. Because it is the middle score, the median is the 50th percentile.

Here's an example. Seven basketball players shoot 30 free throws during a practice session. The numbers of baskets they make are listed below. What is the median number of baskets made?

To find the median, use the following steps:

1. Put the scores in ascending or descending order. If you do not first do this, the median will merely reflect the arrangement of the numbers rather than the actual number of baskets made. Here are the scores in ascending order.

11, 15, 18, 20, 22, 22, 23

2. Count in from the lowest and highest scores until you find the middle score.

What is the median number of baskets? The median number of baskets is 20 because there are three scores above 20 and three scores below 20.

Here's another example. Twelve members of a gym class, some in good physical condition and some in not-so-good physical condition, see how many sit-ups they can complete in a minute. Here are their scores.

2, 3, 6, 10, 12, 12, 14, 15, 15, 15, 24, 25

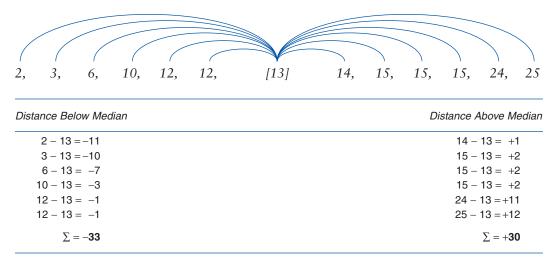
What is the median number of sit-ups? Is it 12? 14? The median is 13, because there are six scores below 13 and six scores above 13. Note that the median does not necessarily have to be an existing score. In this case, no one completed exactly 13 sit-ups.

Here is the rule: With an odd number of scores, the median will be an actual score. But with an even number of scores, the median will not be an actual score. Instead, it will be the score midway between the two centermost scores. To get the midpoint, simply average the two centermost scores. In our example, this is (12 + 14)/2, which is 26/2, which is 13.

While the number of scores on each side of the median must be equal, the numerical distance of the scores on either side of the median will not necessarily be equal. It might be equal, but it doesn't have to be. Finding the distance of each score from the median, we get the following values. As shown in boldface, -33 does not balance +30.

Q: Where does a statistician park his car?

A: Along the median.



One nice feature of the median is that it can be determined even if we do not know the value of the scores at the ends of the distribution. In the following set of seven pop quiz scores (oops—it looks like the students weren't prepared!), we know that there is a score above 70 but do not know what that score is. Likewise, we know that there is a score below 30 but not what that score is:

Nevertheless, we can determine the median by counting up (or down) half the number of scores. In this case, the median is 50, because it is the fourth score from either direction. It does not matter whether the top score was 90, 100, or even 1,076 or whether the bottom score was 20, 10, or even -173. The median is still 50.

It is also possible to compute a median from a large number of scores when there are many duplicate scores. Suppose the pop quiz were given not just to 7 students but to 90 students. Because of the large number of students at each score, it is easier to interpret the data if they are arranged in a frequency table. Table 5.1 gives the scores and their frequencies.

Score	Frequency	Cumulative Frequenc	
>70	3	90	
70	7	87	
60	19	80	
50	31	61	
40	14	30	
30	12	16	
<30	4	4	
	$\Sigma = 90$		

Table 5.1Pop-Quiz Scores for 90 Students

There are 90 scores in all. Thus, the median will have 44.5 scores above it and 44.5 scores below it. To get an estimate for the median, start at the bottom and count upward: 4 + 12 = 16 cases (proceed); 16 + 14 more = 30 cases (proceed); 30 + 31 more = 61 cases (stop). The score of 50 is the median because we reach the middle case at that score.

Because there are many cases at each score, a reading from a frequency table gives, as we saw in Module 3, only a ballpark figure. Because the median is the 50th percentile, the median score out of 90 cases should be the score of the 44.5th case out of the 90 cases. But note that there are only 30 cases below our ballpark median of 50 (14 + 12 + 4), not 44.5 cases. Thus, if we take the bottommost case of the students who scored 50, that person's score is the 31st case from the bottom, and if we take the topmost case of the students who scored 50, that person's score is the 61st case. Obviously, the 44.5th case falls somewhere within the 31 students who scored 50. We already know that 30 students scored below 50; thus, we need only 14.5 additional cases of the 31 cases at the score of 50 to reach the 44.5th case.

Settling for 50 as the median is like throwing darts at a dartboard. If it hits anywhere in the bull's-eye, we say we've hit the bull's-eye. But even within the bull's-eye, some points are more central than others. To determine the exact bull's-eye, we'd need to get out a measuring instrument and find the precise center of the bull's-eye.

And so it is with the median. For precision, we must use a formula. Here is the formula for a median:

$$Mdn = LL + (i) \left(\frac{0.5n - \operatorname{cum} f_{\text{below}}}{f}\right)$$

where

LL = lower real limit of the score containing the 50th percentile,

i = width of the score interval,

0.5n = half the cases,

cum f_{below} = number of cases lying below the LL, and

f = number of scores in the interval containing the median.

First, we determine the LL of the score containing the median. Recall that the real limits of a score extend from one half the unit of measurement below the score to one half the unit of measurement above the score. In our quiz example, the unit of measurement is 10 points; that is, scores are expressed to the nearest 10 points (40, 50, 60, etc.). Therefore, the real limits of a score are ± 5 points. Table 5.2 is the frequency table with scores reexpressed in real-score limits.

Score	Real Limits	Frequency
>70	>75	3
70	65-75	7
60	55–65	19
50	45–55	31
40	35–45	14
30	25–35	12
<30	<25	4
		$\Sigma = 90$

 Table 5.2
 Real Limits of Pop-Quiz Scores for 90 Students

We already determined that the median falls in the score interval of 45 to 55. The LL of that interval is 45. Plugging the LL into the formula, we get the following:

$$Mdn = LL + (i) \left(\frac{0.5n - \operatorname{cum} f_{\text{below}}}{f}\right)$$
$$= 45 + (10) \left(\frac{(0.5)(90) - 30}{31}\right)$$
$$= 45 + (10) \left(\frac{45 - 30}{31}\right)$$
$$= 45 + (10) \left(\frac{15}{31}\right)$$
$$= 45 + (10)(0.4838)$$
$$= 45 + 4.838$$
$$= 49.838$$

The ballpark median was 50, but the exact mathematical median is 49.838. The mathematical median is a bit different from what we proposed by counting up cases from the bottom. Why? Remember that we needed only 14.5 of the 31 cases at score 50 to bring our case count up to 44.5. Out of 31 cases, 14.5 is just less than half. Remember also that the real limits for a score of 50 are 45 and 55. If we assume that the 31 scores at the score of 50 are evenly distributed between 45 and 55 and we go just less than halfway into the 45 to 55 range, what do we get? We get a little less than 50—or 49.838, our calculated median!

PRACTICE

1. One hundred and thirty-six breast cancer survivors participate in a community walk to raise money for fighting the disease. The number of women who walked various numbers of miles is listed below:

Miles Walked	Number of Women
5	18
4	23
3	54
2	19
1	8

a. Counting up from the bottom of the table, what is the ballpark median number of miles walked?

- b. Using the formula for a median, find the exact median number of miles walked by these women.
- 2. Morbidly obese women attending the Healthy Weigh diet clinic are weighed at program entry, to the nearest 10 lb. To join, the women must weigh at least 200 lb. Here are the women's weights.

Number of Women
1
4
0
4
9

Weight (lb)	Number of Womer
240	10
230	14
220	18
210	11
200	9

- a. Counting up from the bottom of the table, what is the ballpark median weight of clinic clients?
- b. Using the formula for a median, find the exact median weight of clinic clients.
- 3. Fifty clients of an outpatient mental clinic take an anxiety inventory. Scores range from 1 to 10. Here are the scores.

Anxiety Score	Number of Client	
10	2	
9	8	
8	13	
7	10	
6	7	
5	4	
4	2	
3	2	
2	1	
1	1	

- a. Counting up from the bottom of the table, what is the ballpark median anxiety score?
- b. Using the formula for a median, find the exact median anxiety score.
- 4. Twenty autistic children in a communication therapy program are scored on the number of times in a given session that they initiate eye contact or direct a comment toward the primary caretaker. Here are their scores.

Number of Contacts	Number of Children
8	1
6	1
5	1
4	2
3	4
2	7
1	4

- a. Counting up from the bottom of the table, what is the ballpark median number of contacts?
- b. Using the formula for a median, find the exact median number of contacts.

Mean

The mean, symbolized M (for samples) or μ (for populations), is the average score. You already know how to calculate an average. If you want to know the average score on a class test, you add up all students' scores and divide by the number of students in the class, right? In statistics, we make that process explicit with a formula. Here is the formula for a mean:

$$M = \frac{\sum X}{N}$$

where

 $\Sigma =$ sum, X = raw score, and N = number of cases.

The formula says to add up (Σ) all the raw scores (X) and divide by the number of cases (N). Whenever you divide anything by the number of cases, you get an average of the thing you were dividing. The "thing" in this case is raw scores. Thus, the formula gives the average raw score.

Of the three measures of central tendency, the mean is the most stable. That is, if we drew many samples of the same size from the same population and calculated the mean of each sample, the mean would not likely vary much from sample to sample.



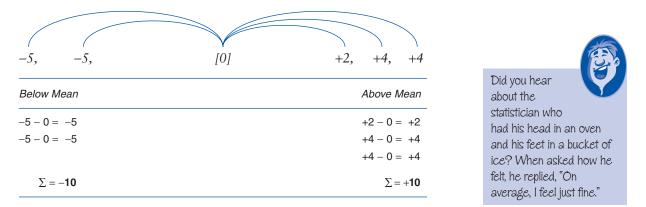
"The poor are getting poorer, but with the rich getting richer it all averages out in the long run."

SOURCE: Joseph Mirachi, New Yorker Cartoon 9/26/1988 © Joseph Mirachi/Condé Nast Publications/www.cartoonbank.com

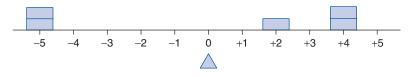
Most important, the mean is the place where the numerical distances of scores on one side of the mean balance the numerical distance of scores on the other side of the mean. To demonstrate this principle, first find the mean of the following Fahrenheit temperatures for five winter days in Maine: -5, -5, +2, +4, +4. You should be able to do it in your head. The mean is 0:

$$M = \frac{\sum X}{N} = \frac{(-5) + (-5) + (+2) + (+4) + (+4)}{5} = \frac{0}{5} = 0$$

Finding the distance of each score from the mean, we get the following values. As shown by the boldface in this example, -10 exactly balances +10.



Let's place these scores on a balance board such as a teeter-totter or seesaw. Imagine that each block is a score, and the scores are placed at distances equal to their numeric values. There are two scores of -5, one score of +2, and two scores of +4. As you can see in Figure 5.1, the scores balance at the mean, which in this case is 0.





The teeter-totter demonstrates other important characteristics of the mean. First, there do not necessarily have to be the same number of scores on each side of the mean. In this case, there are two scores below the mean and three scores above the mean.

Second, the numeric value of each score, X, is included in the calculation of the mean. Thus, the value of each score matters. The mean is not dependent on score frequencies, as was the case with the mode. Nor is the mean dependent on position within a distribution, as was the case with the median. But the mean is dependent on the value of each score.

This leads to a third important characteristic of the mean. Because the value of each score matters in the calculation of the mean, the mean is the most sensitive of the measures of central tendency to score aberrations. That is, a single extreme score has a marked effect on the mean's value.

To demonstrate this, let's consider the preceding set of temperatures again but replace one of the +4 temperatures with a temperature of +34. The new set of temperatures is -5, -5, +2, +4, +34. Here is the new mean. It has jumped from 0 to 6.

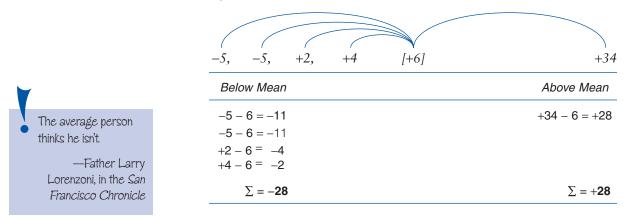
$$M = \frac{\sum X}{N} = \frac{(-5) + (-5) + (+2) + (+4) + (+34)}{5} = \frac{30}{5} = 6$$

Let's put this new set of scores on the teeter-totter as well. The break in the line to the right of Figure 5.2 indicates that a portion of the teeter-totter is missing. Without the break, the score of +34 would be somewhere off the right-hand side of the page.



Figure 5.2 Scores Balancing at Mean of +6

Because of the increase in the mean, a single score above the mean now balances four scores below the mean, as shown in boldface below.



HERMAN®



"How do you expect me to average 55 miles an hour if I don't speed?"

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A score that is way out of line with the rest of the data is called an outlier. Sometimes outliers are legitimate—one person in the sample is simply much faster, smarter, or better along whatever scale is being measured. Other times an outlier represents a clerical error-the person was measured incorrectly or the score was entered into the data set incorrectly. Because outliers markedly affect the mean, researchers need to be especially alert for them so that they can determine whether the score legitimately belongs in the data set. Simply knowing the value of the mean does not, in itself, tell us that there is an outlier. Only visual inspection of the data tells us that. This is another reason why competent researchers always look at the data before calculating any statistic.

If there are several outliers, the median is a more appropriate measure of central tendency to report than the mean because the median is not influenced by outliers. In most cases, however, the mean is the preferred measure of central tendency to report. This is because further statistical analyses build on the mean. Sample means, for example, play an important role in the population-based statistics found throughout the remainder of this textbook.

CHECK YOURSELF!

	Mode	Median	Mean
Ease of calculation			
Stability over time			
Distance of scores above and b	pelow		
Number of scores above and be	elow		

PRACTICE				
5. Here, again,	is the set of statistics	s test scores we worked	with in Modules 3 a	nd 4.
98	89	86	84	76
96	89	86	84	76
94	89	86	83	75
94	88	85	83	73
93	88	85	82	73
92	88	85	82	72
92	87	85	80	69
91	87	84	80	66
90	87	84	79	62
90	87	84	79	56

Find the (a) mode, (b) median (by formula), and (c) mean for these data.

6. Exercise 2 in Module 3 gave the following number of classes each of 32 students missed during the semester for a class meeting on M-W-F.

0, 1, 4, 2, 8, 1, 2, 4 5, 0, 2, 2, 1, 2, 1, 2 1, 0, 3, 1, 1, 2, 1, 2 2, 4, 0, 1, 3, 0, 3, 3 Find the (a) mode, (b) median (by formula), and (c) mean for these data. 7. Exercise 3 in Module 3 gave the following number of pets owned by customers of a pet store.

3, 0, 1, 4, 3, 2, 2, 1, 3, 0, 2, 4, 5, 3, 2, 4, 7, 1, 1, 2

Find the (a) mode, (b) median (ballpark), and (c) mean for these data.

8. Exercise 4 in Module 3 gave the following number of TV sets in homes.

2, 3, 1, 2, 3, 0, 2, 4, 1, 2, 4, 3, 2, 1, 1, 3, 0, 2, 1, 1, 2, 3, 2, 5, 2

Find the (a) mode, (b) median (ballpark), and (c) mean for these data.

Skew and Central Tendency



Three statisticians went target shooting. The first one took aim, shot, and missed by a foot to the left. The second one took aim, shot, and missed by a foot to the right. Whereupon the third one exclaimed, "We got it!" and walked away. Recall from Module 4 that skew is a measure of asymmetry in a set of data. Skew affects the location of the mode, median, and mean. In a symmetric distribution such as a normal distribution, the three measures of central tendency coincide. That is, the most frequent score (mode) equals the midpoint (median), which equals the average (mean) (Figure 5.3).

This is not the case in a skewed distribution. Scores in the tail of a skewed distribution are outliers. And we already saw via the teeter-totter what happens when an outlier is introduced: The mean moves in the direction of the extreme score—that is, toward the tail. Because the mean is the most sensitive of the three measures of central tendency to extreme scores, the mean is pulled most toward the tail. The mode, which is simply the most frequent score, remains where it was. The median falls between the mean and the mode. This

happens in both negatively and positively skewed distributions (Figure 5.4).

Because of the known relationship of the mode, median, and mean in normal versus skewed distributions, a researcher can tell from the calculated values whether a distribution is normally distributed or skewed. From Figures 5.3 and 5.4, we see that if the mean is lower than the mode, the distribution is negatively skewed. Conversely, if the mean is higher than the mode, the distribution is positively skewed. Similarly, a researcher can tell from the shape of the distribution where the mean, median, and mode will fall. If a distribution is negatively skewed, the mean must be lower than the mode. Conversely, if a distribution is positively skewed, the mean must be higher than the mode.

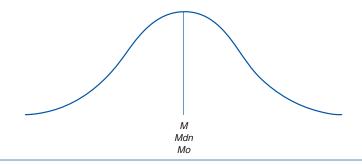


Figure 5.3 Position of Mean, Median, and Mode in Normally Distributed Data

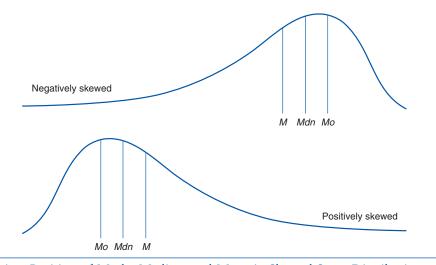


Figure 5.4 Position of Mode, Median, and Mean in Skewed Score Distributions

Comparing distribution shape with central tendency values is another way in which researchers check for clerical errors as they analyze their data. Minor deviations from expectation (say, a median that exceeds the mode or is lower than the mean) are usually due to "lumpiness" (additional lesser modes) in the data. But if the graphs and the numbers differ markedly, there is probably a calculation error.

Putting all this information together, which measure of central tendency should we report for a given set of data?

- Always report the mean if it is appropriate to do so, because it is the most useful measure for further statistical analysis.
- If we don't know the exact value of every score, we cannot report the mean. We may, however, be able to report the median.
- The median is a better choice when a distribution is known to be seriously skewed. In that case, the mean would be misleading.
- The mean is not appropriate for describing a bimodal or multimodal distribution. This is shown in Figure 5.5, in which the mean seriously misrepresents both the higher and the lower clusters of scores.

Note that multimodality is another instance in which a graph tells a story that a summary statistic cannot. As you can see from the graph, when a sample consists of two or more subgroups with very different performance levels, reporting any single measure of central tendency seriously misrepresents the performance of either group. In such a case, it is best to report two modes, one for each group.

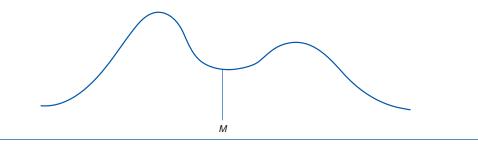


Figure 5.5 Position of the Mean in a Bimodal Distribution

PRACTICE

- 9. In Exercise 3 in this module, you found the median anxiety score for 50 mental health clinic clients.
 - a. Is the set of anxiety scores normally distributed, positively skewed, or negatively skewed?
 - b. Based on the distribution shape, would you expect the mean to be higher or lower than the median?
- 10. In Exercise 4 in this module, you found the median number of contacts made by 20 autistic children.
 - a. Is the number of contacts normally distributed, positively skewed, or negatively skewed?
 - b. Based on the distribution shape, would you expect the mean to be higher or lower than the median?
- 11. In Exercise 5 in this module, you found the mode, median, and mean for a set of 50 statistics test scores. Table 3.7 in Module 3 presents a grouped frequency table for that same data.
 - a. Is the set of scores normally distributed, positively skewed, or negatively skewed?
 - b. Do the relative magnitudes of the mode, median, and mean agree with the shape of the distribution? If not, can you explain why not?
- 12. In Exercise 6 in this module, you found the mode, median, and mean for an attendance log for a college class. In Exercise 2 in Module 3, you created a frequency table for that same data.
 - a. Is the set of scores normally distributed, positively skewed, or negatively skewed?
 - b. Do the relative magnitudes of the mode, median, and mean agree with the shape of the distribution? If not, can you explain why not?
- 13. You plan to try out for a track team. Every morning for a month, you run a mile and time yourself. What measure of central tendency best summarizes your running speed for the month? Why?
- 14. Your instructor gives a 60-item pretest on the first day of your statistics class. Because there has not yet been any statistics instruction, most students score quite low on the test. However, a small number of students have had statistics instruction as part of another course. Those students do very well on the test. What measure of central tendency best summarizes the whole class's statistics knowledge at the start of the course? Why?
- 15. A social psychologist wants to know the physical distance at which people are comfortable when in face-to-face conversation. She engages 20 participants in conversation on a neutral topic. Later, using a stationary video of the conversation taken against a 1 in. \times 1 in. background pattern, she measures the number of inches each person stood from her when speaking. What measure of central tendency best summarizes the speaking distance of the 20 participants? Why?
- 16. A college's health center sees about 50 students per week. Most visits fall into one of these categories, in order of frequency: (1) infectious diseases, such as strep throat, mononucleosis, and influenza; (2) muscle, tendon, or ligament sprains and strains; or (3) complications of existing chronic disorders such as asthma or diabetes. It is the policy of Health Center staff to follow up with each patient after an office visit until the patient reports that his or her health has returned to normal. What measure of central tendency best summarizes patient recovery time for all patients seen during a typical week? Why?

SPSS Connection

Download the file data_score set for central tendence.sav from www.sagepub.com/steinberg2e. These data are used in the textbook example.

Alternatively, manually enter the following five scores into the SPSS Data View spreadsheet: -5, -5, +2, +4, +4. Click on the Variable View tab to define the variable. Name the variable scr4mean, set the decimals at 0, and label the variable as Scores.

If the file is not already in Data View, click that tab in the lower left of the screen.

In the toolbar at the top of the screen, click on Analyze, then Descriptive Statistics, then Descriptives. Highlight the variable Scores in the left window and click on the arrow between the windows to send that variable into the right window. Click on Options. Several boxes are already checked by default. Keep the check mark in the box for Mean. Remove checkmarks in the three boxes in the Dispersion section by clicking on them. Click Continue and then OK. This is what you will see.

Descriptives

Descriptive Statistics

	Ν	Mean
Scores	5	.00
Valid N (listwise)	5	



Visit the study site at www.sagepub.com/steinberg2e for practice quizzes and other study resources.